

ON METHODS FOR THE CONSTRUCTION OF NETWORKS DUAL TO NON-PLANAR NETWORKS

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ABSTRACT. Two networks are said to be dual to each other if the following correspondence exists between them :—

- (a) Each element of one network has a counterpart in the other one.
- (b) The currents through the elements in one network and the voltages across their counterparts in the other network are proportional to each other.

A network dual to a given one is usually constructed by connecting elements of the appropriate kind in a circuit the diagram of which forms the topological counterpart to the diagram of the original circuit. If this original circuit is a "non-planar" one (i.e. if it cannot be drawn without crossings between some of its branches), then there exists no topological dual, and for this reason it has been said that there exists no dual to a non-planar electrical network. However, if we are interested in an electrical network not because its circuit diagram has certain geometrical properties, but because it constitutes an assembly of a number of passive and active elements which perform in a certain manner (on account of their impedance properties and of certain relations of mutual constraint imposed by Kirchoff's mesh and junction relations), then there is really no restriction of this kind.

We need only convert the original network into another with the same elements and the same performance, but with a planar circuit diagram and with additional constraints implemented by ideal transformers. This equivalent network can be then converted into its dual counterpart, which is another planar network subject to the dual counterparts of the transformer constraints of the first case. In this way we arrive at an assembly which performs in the required dual manner.

Introducing thus the concept of physical duality as opposed to geometrical duality, the paper explains in an elementary manner a number of ways in which this conversion into equivalent planar networks can be achieved. These methods are :—

- (1) A method originally published by Julia.
- (2) The dual counterpart of this method.
- (3) The *method of fictitious junction points* (an electrical analogue to F. Schur's method of fictitious junction points in pin-jointed frameworks).
- (4) The *method of the transferred terminal*.
- (5) The general *method of separation*.

Some of these methods require a smaller number of ideal transformers than Julia's original method.

The paper ends with a short discussion of the errors introduced when the dual network is set up on a network analyser and when the ideal transformers have to be replaced by real ones ; the numerous methods available for the realization of an equivalent network enable a choice to be made so as to minimize these errors, or possibly eliminate them altogether.

The existence of a dual to a non-planar network is of some interest in connection with electro-mechanical analogies as it removes a restriction to which the "direct" analogy would otherwise be subjected.

§1. INTRODUCTION: THE PRINCIPLE OF DUALITY

TWO networks are said to be dual to each other if the following correspondence exists between them:—

- (a) Each element of one network has a counterpart in the other.
- (b) The currents through the elements of one network and the voltages across their counterparts in the other network are proportional to each other.

In other words, we have the following relations:—

$$i_{\kappa'} R_0^1 = e_{\kappa''}, \quad \dots\dots(1)$$

$$i_{\kappa''} R_1^0 = e_{\kappa'}, \quad \dots\dots(2)$$

where $i_{\kappa'}$ and $e_{\kappa'}$ are current and voltage at element κ' of the first network and $i_{\kappa''}$ and $e_{\kappa''}$ current and voltage at the corresponding element κ'' of the second network, and where R_0^1 and R_1^0 are two constants (each of the dimension of a resistance) which are characteristic of the particular correspondence under consideration.

It is obvious then that the behaviour of one network is a copy of that of the other network, but with the rôles of current and voltage interchanged.

We shall see in the following that the existence of such a relation can be predicted by very simple criteria, and for this reason it will obviously be useful, for it will enable us to predict the performance of one network without a preceding analysis provided we know the behaviour of its dual counterpart. This *Principle of Duality* has been known for some time. The phrase seems to have been introduced into electric network theory by H. Sire de Vilar (1901). The first textbook dealing with it was A. Russell's *Theory of Alternating Currents* (1904), which gives a large number of applications and a list of earlier references. There are, or rather were, however, certain difficulties in the application of the principle whenever one of the networks concerned was "non-planar."*

These difficulties give rise to an interesting question in connection with electro-mechanical analogies (Bloch, 1945). There are two types of such an analogy: one in which we represent a force by a voltage (the "direct" analogy), and one in which we represent a force by a current (the "inverse" analogy). If we represent a given mechanical system once by the first and once by the second type of analogy then we obtain two electrical networks which are—obviously—in dual relationship to each other. Now the circuit diagram of the inverse type of analogy can always be obtained as a simple translation of the mechanical layout if this is drawn according to certain conventions. Should this circuit diagram be of the non-planar type, and were we to take the statement literally that there exists no dual to such a network,† then we would have to acknowledge that the field of application of the direct analogy was restricted. We shall see that this is not the case, and that the apparent contradiction turns on the definition of the concept of duality. Before concentrating on the special

* A non-planar network is one the circuit diagram of which cannot be drawn without crossings between some of its branches.

† Whitney (1931) shows that there exists no geometrical dual to a non-planar graph; Barnes and Gardener (1932) seem to take this as the basis of their statement that there exists no dual to a non-planar network.

problem of the present paper we shall therefore deal first more fully with the normal and general aspects of the duality principle.

It will be seen that we have to distinguish between duality in a topological sense and duality in a physical sense. By duality in a physical sense we have in mind a correspondence between two assemblies of elements such as outlined above, a correspondence which is concerned only with the behaviour of the elements specified. By duality in the topological sense we have in mind a relation between the structure of the two circuit diagrams such that each branch of one network has its counterpart in the other network and that all those branches of one network which radiate from a particular junction point have counterparts in the other network which form a closed mesh. The fact that the branches of the network are linked together in a certain way imposes on the currents therein and on the voltages across them certain conditions of constraint commonly known as Kirchhoff's equations. These conditions of constraint are obviously decisive for the behaviour of the network. Now we shall see that in order to bring about dual behaviour of two networks it is sufficient to establish a certain impedance relation between corresponding elements of the two networks, to provide external sources of dual character, and finally to provide conditions of constraint which are of dual character. If we insist in bringing about this dual type of constraint by an appropriate structure of the circuit diagram—and by this means only—then we shall see that the concepts of physical and of topological duality are inseparably linked. On the other hand, if we are only interested in the behaviour of the networks (i.e., assemblies of elements), such topological duality is immaterial; we may then introduce the required conditions of constraint by other means. These other means are ideal transformers, and we shall see by their use it will always be possible to specify one or more networks which are dual in the physical sense. Thus the theorem that there exists no topological dual to a non-planar network does not occupy the dominant position which one might at first assume. It is nevertheless still of interest, for obviously topological duality is the simplest way of bringing about dual conditions of constraint, and we shall always try first to achieve it. Knowledge of the theorem will, however, save us from useless attempts to find this ideal solution.

The present paper discusses first R. Julia's method for deriving a physically dual network by the introduction of ideal transformers. By giving the justification for this procedure more fully than it was given in his original publication we are automatically led to further methods of achieving the same result. This is useful, for some of the new methods require a smaller number of ideal transformers than Julia's original method. At the same time the fact that we are thus able to choose between a whole series of different embodiments of the dual network will be useful when we attempt to set up the dual network in a network analyser, for some of these alternative ways will possibly be less susceptible to the imperfections introduced by the use of real transformers in place of the ideal ones.

§2 CRITERIA OF DUAL CORRESPONDENCE: CONSTRUCTION OF THE DUAL COUNTERPART IN THE CASE OF PLANAR NETWORKS

The existence of the dual relationship between two networks as specified by equations (1) and (2) can be predicted very simply if we pause for a moment

and consider what data determine the behaviour of a given network. There are:—

- (a) The relations between the currents through and the voltages across the various elements of the network, i.e. a series of equations of the type

$$i_{\kappa'} = f(e_{\kappa'}); \quad \dots\dots(3)$$

- (b) The way in which these elements are connected, i.e. the relations of constraint which exist between the currents meeting at a given junction point and the voltages which we encounter when we pass around a closed mesh (Kirchhoff's equations);
- (c) The initial state of the network (currents through and voltages across certain elements);
- (d) The external "causes" (voltages or currents) impressed on the network.

All these data together give us a system of equations—not necessarily a linear one—the solution of which describes the behaviour of the network.

Suppose now that we have a second network for which we can list a corresponding set of data, the correspondence being of the type described by equations (1) and (2). This means that we get a corresponding set of data which is of the same form as the original one but with the difference that the symbols i and e have changed places. It is obvious then that the resultant system of equations will show the same interchange of symbols, and the same holds for the final solution of this system. Hence the second network must show a behaviour dual to that of the first network.

Let us now discuss in detail what is involved in having such an interchanged set of data between two networks. In order not to interrupt the exposition we shall deal first with the items (c) and (d). The dual counterpart of the initial conditions is self-evident; any statement of this type can be translated in a straightforward manner by the application of equations (1) and (2). The same applies to the specification of the external "sources" which are connected to the network. A voltage source corresponds to a current source of proportional strength and vice versa.

As regards (a), equation (3) is, according to equations (1) and (2), replaced by an equation of the type

$$\frac{e_{\kappa''}}{R_0^1} = f(i_{\kappa''} R_1^0), \quad \dots\dots(4)$$

i.e., apart from factors of proportionality, the graph of current against voltage for each element of the original system will represent the graph of volts against current in the new system. If we limit ourselves to linear systems and study the steady-state behaviour of such systems at a frequency ω we have for any particular element of the first system and for its counterpart the relations

$$e_{\kappa'} = i_{\kappa'} Z_{\kappa'} \quad \text{and} \quad e_{\kappa''} = i_{\kappa''} Z_{\kappa''}, \quad \dots\dots(5)$$

where $Z_{\kappa'}$ and $Z_{\kappa''}$ denote the impedances of the elements concerned. Multiplying these two equations and taking into account equations (1) and (2), we obtain

$$R_0^1 R_1^0 = Z_{\kappa'} Z_{\kappa''}. \quad \dots\dots(6)$$

Equivalent expressions for the last equation are

$$\left. \begin{aligned} Y_{\kappa}' &= Z_{\kappa}'' \frac{1}{r_0^2}, \\ Y_{\kappa}'' &= Z_{\kappa}' \frac{1}{r_0^2}, \end{aligned} \right\} \dots\dots(7)$$

where $Y_{\kappa}' = 1/Z_{\kappa}'$ and $Y_{\kappa}'' = 1/Z_{\kappa}''$,

and where $r_0^2 = R_0'R_1^0$(8)

From these equations it follows that the dual counterpart of a resistance R is a conductance $G = R/r_0^2$, that the counterpart of an inductance L is a capacitance $C = L/r_0^2$ and that of a capacitance C an inductance $L = Cr_0^2$.

It will be noted that these relations are independent of the frequency ω , i.e., that dual relationship once found for certain frequencies will be valid for all frequencies. On account of Fourier's theorem this means that once the dual relationship is established for steady-state conditions at one particular frequency it will hold good not only for steady-state operation at any other frequency but also for transient behaviour.

Turning now to point (b), the problem of interconnection of elements, we will discuss a special case first, namely the case where two or more elements in the first system are connected in series. This means that all these elements carry the same current, and that their voltages add. Their dual counterparts must then be so connected that across each of them appears the same voltage, and that all their currents add, i.e. they must be connected in parallel. This relation works, of course, the other way round. It is a very useful relation, for it can be applied for checking even when we have to deal with a more general type of network; we can always give to some of the impedances of a network the values zero or infinity, and thereby derive from the given network a number of simpler networks to which these conversion rules can be applied. (We might establish here a principle of "detailed reciprocity", the name being chosen in analogy to the "principle of detailed balancing" as used in thermodynamics).

A more general way of establishing relations of constraint dual to the given ones is based on a consideration of Kirchhoff's equations. Suppose we have for a junction point (σ) of the given network written down the condition that the sum of all branch currents reaching this point is zero, i.e.

$$i'_{\sigma_1} + i'_{\sigma_2} + \dots = 0. \quad \dots\dots(9)$$

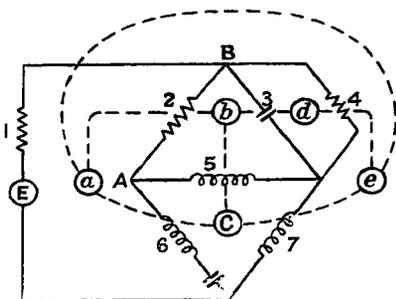
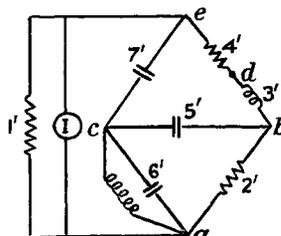
It follows then that in the dual network the voltages across the counterparts of all these branches must add up to zero:

$$e'_{\sigma_1} + e'_{\sigma_2} + \dots = 0, \quad \dots\dots(10)$$

i.e., we would normally expect these counterparts to be so arranged that they formed a closed mesh. Conversely we see that elements which are so arranged that they form a closed mesh should have dual counterparts which are joined to a common junction. We see that dual conditions of constraint are automatically introduced if the second network is in this particular geometrical relation to the first one, i.e. if it is the topological dual of the first network.

There exists a routine method for drawing such a topological dual (Cauer, 1934).

In the interior of each mesh of the given network we select a point. These points will form the junction points of the new network; we need only join each of these points to its neighbours (i.e. to those new points situated in adjoining meshes) in order to complete the layout of the dual network. Each new branch crosses one old branch (the branch which was common to the two adjoining meshes). The procedure is illustrated in figures 1 *a* and 1 *b*. In figure 1 *a* the new circuit is sketched in dotted lines, in figure 1 *b* it is redrawn, and at the same time the proper elements inserted, taking into account the relations (6) and (7) and their consequences as to the interchange of capacities and inductances. It is seen that in a case where the original branch is composed of several elements in

Figure 1 *a*.Figure 1 *b*.

series (e.g. 4 and 6) we could draw such a crossing line through each of these elements; all these new branches are then in parallel, and if we place into each of these new branches an element with an admittance proportional to the impedance of the "crossed" old element the resulting compound admittance will obviously be the correct dual of the original compound impedance.*

The procedure just described breaks down if the original circuit diagram cannot be drawn without crossings between some of the lines ("non-planar" networks). Such networks have no topological duals, and in their case we have to use other means for introducing the required constraints between the various elements of the dual network.

§ 3. IDEAL TRANSFORMERS

We shall now enter into a discussion of these other means which will enable us to introduce the required constraints. Suppose we have an iron core with several windings of the same number of turns, i.e. a multi-winding transformer, schematically represented in figure 2 *a*. Let us assume further that the permeability of the core material is infinite and that the windings have negligible leakage. In this case we are then assured that the same flux is linked with each of the windings, i.e. that the terminal voltages of all the windings are equal. Furthermore, the finite flux required for any such finite voltage can then be

* It will be noted that this method of drawing the dual circuit can also be used to define in a consistent manner the sense of the dual currents or voltages. We need only agree that the arrows defining this sense in the new branches must be produced from those of the original network by, say, a rotation in a clockwise rotation.

produced by a magnetomotive force which is infinitesimally small, i.e. we must have

$$i_1 + i_2 + i_3 + \dots = 0. \quad \dots\dots(11)$$

A transformer of this kind is called an *ideal transformer*. Real transformers differ from the ideal one by possessing leakage inductance which we might visualize as a separate inductance connected in series with each individual winding and by requiring a finite magnetomotive force for the creation of their magnetic flux. With real transformers the voltages across the individual windings may then no longer be exactly equal, and furthermore, equation (11) will be only approximately fulfilled. These deviations are, however, usually small, and they can possibly be allowed for if it is desired actually to build the dual network. For a discussion of principles we are certainly allowed to make use of the concept of an ideal transformer, and we see that a transformer of the type just described imposes on the currents through its windings the same constraints as if they were associated with a common junction point.

The dual counterpart to such a transformer is shown in figure 2 *b*. There the magnetic circuit is so arranged that the cores of the individual coils—which

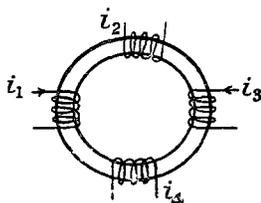


Figure 2 *a*.

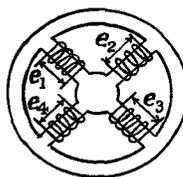


Figure 2 *b*.

in figure 2 *a* were all connected in series—are all connected in parallel. In this magnetic circuit, the sum of all the magnetic fluxes which arrives at one of the magnetic junctions must be zero, and from this it follows that the sum of all the voltages of the windings of the various cores (which individually are proportional to the fluxes of the corresponding cores) must be zero:

$$e_1 + e_2 + e_3 + \dots = 0. \quad \dots\dots(12)$$

Associated with this condition is the condition that the currents through all the windings must be equal; otherwise the magnetomotive force of the individual limbs would not be equal and—in view of the assumed infinite permeability of the core material—the difference in the m.m.f. of any two of the limbs would drive a magnetic flux of infinite magnitude around the closed circuit formed by these two limbs. A transformer of the second type enforces, therefore, the same type of constraint as results from the existence of a closed mesh. Transformers with two windings are obviously special cases of these general multi-winding transformers, and we see that in their case the two types are identical.

Transformers with two windings are easier to build so as to approach the requirements of zero leakage inductance. It is therefore of interest—for those cases where it is intended to set up a physical circuit—that the same results that can be obtained by a multi-winding transformer with *n* windings can be obtained

by the use of $n-1$ transformers with two windings only. If we start with n transformers with two windings we need only connect all the secondaries in series forming a closed loop, in order to enforce on all the primary voltages the condition that their sum is zero. On the other hand, if we connect all the secondaries in parallel we enforce the condition that the sum of all the primary currents is zero. As figure 3 shows (for the case $n=3$), one of those transformers is actually redundant. (For the purpose of derivation it is often convenient to retain n transformers and to carry out the reduction to $n-1$ as a last step.)

Having thus shown that these transformers will enable us to enforce constraints of the "mesh" and "junction" type,* we have to consider how the method, described in the preceding section, for the routine derivation of the dual network can be operated in the presence of such circuit elements. The dual network we would expect to be subject to transformer constraints which are the dual counterparts of the transformer constraints of the original network. It will be seen that this routine method automatically takes care of the situation provided we stipulate that each transformer coil encountered in this process is taken to

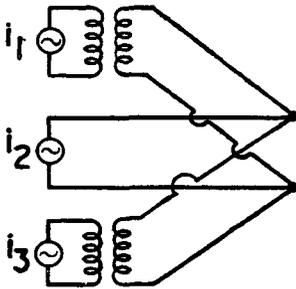


Figure 3 a.

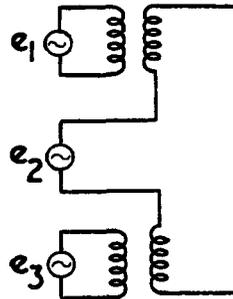


Figure 3 b.

have as counterpart another transformer coil assembled on an ideal transformer of the dual type. Suppose we have in the first network a number of elements Z'_1, Z'_2, Z'_3, \dots , each with such a transformer coil in series, the transformer itself being of the junction type, which insures that the sum of the currents i'_1, i'_2, i'_3, \dots through all these elements is zero. In the new network these coils will automatically be placed in parallel to the dual counterparts Y''_1, Y''_2, Y''_3 of the elements Z'_1, Z'_2, Z'_3, \dots , and thereby enforce the condition that in the dual network the voltages $e''_1, e''_2, e''_3, \dots$ across these elements shall add up to zero. Obviously the method would have worked just as well the other way round, and would have converted a constraint of the mesh type into one of the junction type.

It is perhaps of interest to note that the change over from the one type of transformer to its dual counterpart is actually included in the routine process of "dualization" provided we pay attention not only to the electrical but also to the magnetic circuits, the junction type of transformer having a magnetic circuit dual to that of a mesh type (figures 2 a and 2 b). If we replace the multi-winding transformer by two-winding transformers the exchange of types is simply a consequence of "dualizing" the connection of the secondary windings.

* We shall for this reason refer to such transformers as mesh or junction-type transformers.

§ 4. THE CASE OF NON-PLANAR NETWORKS

(a) *The general principle*

We are now in a position to deal with non-planar networks. Networks of this type can be converted in various ways—presently to be described—into equivalent planar networks which are subject to additional constraints by the aid of one or several ideal transformers. By *equivalence* we mean here that the resultant networks possess, apart from those specially added ideal transformers, the same elements as the original one, and that the currents and voltages at these elements are identical with those of the original network. This equivalent network is then converted into its dual counterpart by the routine method already described.

Of the various methods to convert a non-planar network into an equivalent planar network we shall discuss here the following—without making an attempt to be exhaustive: Julia's method (1939); a method dual to Julia's method; the method of fictitious junction points, the method of the transferred terminal; and the general method of separation.

(b) *Julia's method*

This method is characterized by the following steps:—

(1) Select a suitable junction point and introduce into each of the branches leading to this point the coil of a junction-type transformer. This will not alter the current distribution in the network as the constraint exercised by this transformer is identical with that already in existence. In figure 4 *b* the junction point selected was point A of the original network, illustrated in figure 4 *a*.

(2) Short circuit one of the branches leading to the junction point selected. This has been indicated in figure 4 *b* by the dotted line which connects A to D, and short circuits branch 1. This step again will not alter the current distribution of the network; though we have lifted by this step the constraint originally exercised by the existence of the junction point A, the transformer coils exercise now an identical constraint. Since the sum of the currents flowing through the ideal transformer windings is zero, there can be no current through the short-circuiting link.

(3) Duplicate the short-circuiting link in the way shown in figure 4 *c*, introducing thereby the auxiliary junction points A' and D' (figure 4 *c* shows only those parts of the original network which are essential to the argument).

(4) Cut the link AA'; this cannot alter the current distribution in the network, because the potential difference AA' after the cut is still zero on account of the connection DD'. But now we are allowed to cut DD' as well, for the current through this isthmus must, of course, always be zero.

By these steps we have split the original network into two entirely separate networks, the elements of which have still the same voltages across and the same currents through them as they had before. (We need not have cut DD', for after the cut AA' has been made it would have been possible simply to fold aside the loop containing Z_1 and produce thereby a plane configuration.)

The new plane configuration is redrawn in figure 4 *d*. We may check that on account of the transformer action the current through element Z_1 is still

equal to the algebraic sum of the currents through elements Z_2 and Z_3 ; furthermore, that the voltage drop in going from D to B is equal to the voltages across elements Z_3 and Z_1 , and similarly the voltage drop in going from D to F is equal to drops across Z_2 and Z_1 .

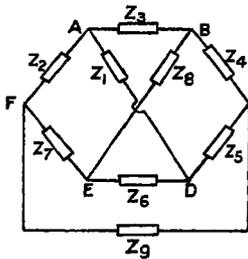


Figure 4 a

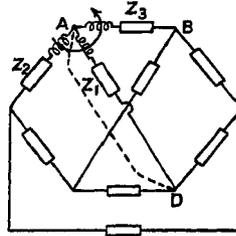


Figure 4 b

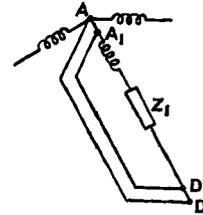


Figure 4 c

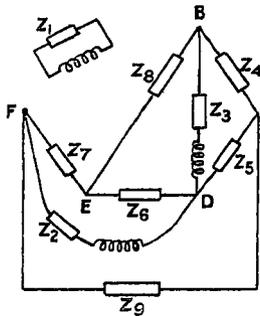


Figure 4 d

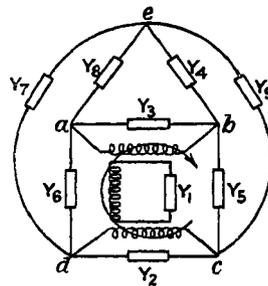


Figure 4 e

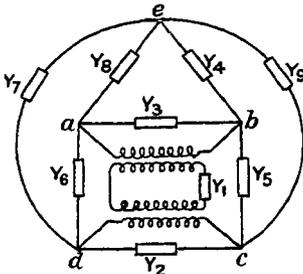


Figure 4 f.

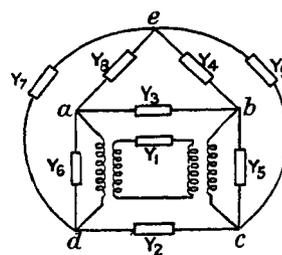


Figure 4 g.

Now the new configuration of elements, being a planar one, can be transformed into its dual counterpart by the method described above. We arrive then at the network shown in figure 4 e. Figure 4 f shows the same network with two 2-winding transformers. Figure 4 g is included for the sake of interest: it shows the final network at which we would have arrived had we eliminated from the original network the junction point D instead of A.

(c) The dual counterpart to Julia's method

We come now to the description of a new method for the derivation of an equivalent planar network which can be characterized as the dual counterpart of Julia's method. In Julia's method we used transformer action to replace a junction type of constraint. Consequently, the dual network arrived at is one in which transformer action is used to implement a mesh type of constraint.

Obviously it is possible to invert this procedure, i.e. we can start by replacing a mesh type of constraint by transformer action and finish them with a dual network in which transformers implement a junction type of constraint. To

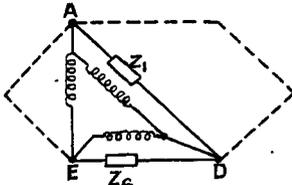


Figure 5 a.

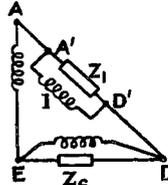


Figure 5 b.

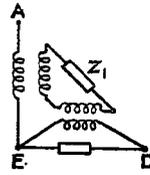


Figure 5 c.

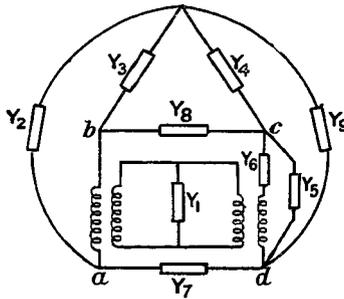


Figure 5 d.

illustrate this procedure we shall carry it out with the same network which we used just now, and we shall again eliminate from this network branch 1. To achieve this we proceed in the following steps:—

- (1) Select a suitable mesh which contains the branch to be eliminated. The mesh to be selected may also include such branches of the original network as contain elements of infinite impedance. Thus, for instance, we may select in figure 4 a the mesh ADEA, of which the branch AE is of infinite impedance, and therefore not explicitly shown as "branch" in the diagram. Place in parallel to each mesh branch the coil of a mesh type of ideal transformer (or the input coil of an ideal two-winding transformer, the secondaries of which are all connected in series). This step, illustrated in figure 5 a,* cannot upset the current distribution in the network, for the constraint thereby imposed is identical with one already in existence.

- (2) Re-draw this network in the way shown in figure 5 b, introducing thereby the auxiliary junction points A' and D'. Cut the link AA' This further step

* The figure shows only those branches which are relevant to the argument.

is permissible without upsetting the network performance as, on account of the transformer action, the potential difference across coil 1 is maintained independent of any current flowing therein; hence the potential difference AA' after the cut must still be zero.*

(3) Cut the link DD' ; this is permissible now—after the cut AA' has been carried out as the current in the isthmus DD' must be zero.

By these steps we have again removed branch 1 from the original network. Figure 5 *c* shows how the same end can be achieved by the use of 2 two-winding transformers. Here it is immediately obvious that the two transformers supply a potential $AE + ED = AD$ to element 1, and that the current drawn by this element is in fact still drawn (via the primaries) from junction point A and returned to junction point D.

Figure 5 *d* shows the dual to this latter network. There is no need to illustrate separately the dual to the network shown in figure 5 *b*.

(d) *The method of the fictitious junction point*

This method can be considered as a generalization of Julia's method; it will as a rule lead to a smaller number of ideal transformers.

Julia's method was characterized by the short circuiting of a branch—this short circuit being without influence on the current distribution on account of the transformer constraint previously introduced. The new method introduces a short circuit at the cross-over of two or more suitably selected branches. By converting thus all the cross-overs in the network into real junction points, the network ceases to be a non-planar one. The original current distribution is again maintained by the introduction of suitable transformer constraints.

The principle of the method is illustrated in figures 6 *a* to 6 *c*. Figure 6 *a* shows two branches of the original network which cross each other without making contact. In figure 6 *b* this cross-over has been converted into a real junction point (X); the original current distribution has nevertheless been maintained by the insertion of the primary and secondary coil of an ideal transformer into the lead BD. The action of this transformer ensures that as much current leaves the new junction point X through Z_4 as arrives there through Z_2 . Hence none of this current can leave through Z_1 or Z_3 . (The introduction of another transformer into the branches containing the elements Z_1 and Z_3 would of course be permissible, and may in a practical case aid in the maintenance of this state of affairs). Figure 6 *c* finally shows the dual counterpart of the network section shown in figure 6 *b*.

An application of the principle just discussed to the network of figure 4 *a* is shown in figures 7 *a* and 7 *b*. It will be noted that by this method we save—compared with the solutions previously given—one ideal transformer. This saving in a number of ideal transformers is larger still in the case of the network shown in figure 8 *a* which represents the second example given in Julia's paper. In dealing with this network Julia removes the junction point A, and requires

* A somewhat more detailed argument is as follows: As the potential across coil 1 remains unchanged, the current through element Z_1 remains also unchanged. After the cut, this current must be supplied from the transformer coil 1, and for this reason all the other coils must then carry a current equal to this current. This means that at junction point A a current equal to this current is still leaving (into coil AE), and at junction point D a current of equal magnitude is still arriving.

for this purpose either an ideal transformer with four windings or three transformers with two windings. From inspection of figure 8 a it follows immediately

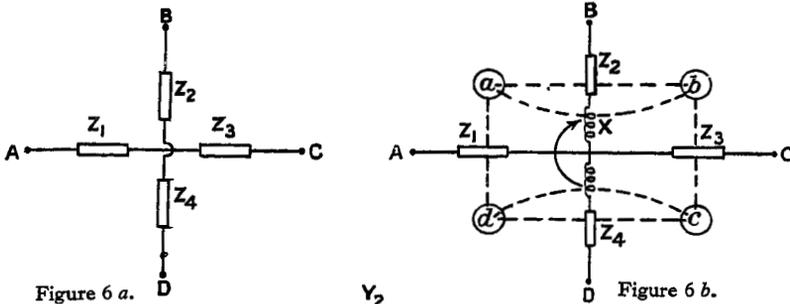


Figure 6 a.

Figure 6 b.

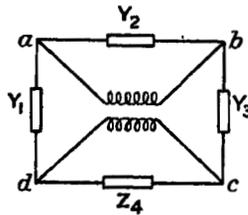


Figure 6 c.

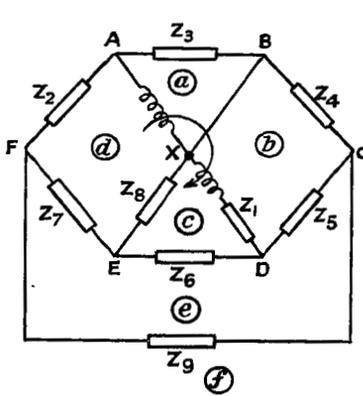


Figure 7 a.

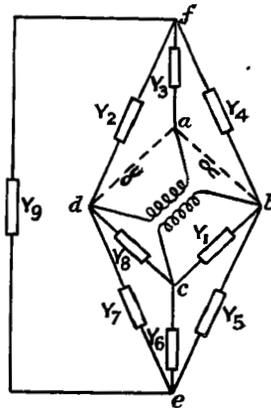


Figure 7 b.

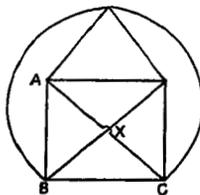


Figure 8 a.

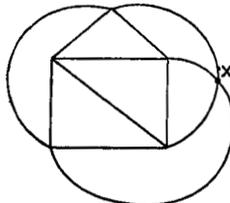


Figure 8 c.

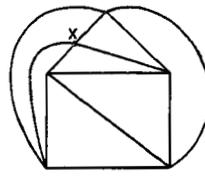


Figure 8 b.

that this case can be dealt with by one ideal transformer with two windings only. Figures 8 b and 8 c show alternative solutions, obviously without exhausting the numerous possibilities.

The artifice of the fictitious junction point here introduced is of some further theoretical interest. As is well known, reciprocal diagrams of the type discussed occur also in graphical statics as reciprocal stress diagrams to pin-jointed frameworks. Frameworks with non-planar lay-out diagrams could not be solved by the original method. This difficulty was overcome by F. Schur (1897) by the introduction of fictitious pin-joints. Whilst, however, such a fictitious joint may be introduced into the lay-out of the framework without upsetting the stress distribution, the application of the analogous artifices in the case of the electrical network requires the special measures discussed above.

(e) *The method of the transferred terminal*

Certain cases of this method arise from an application of the method of the fictitious junction point just discussed. It will be noted that in the example of figure 7 the effect of introducing the fictitious junction point is that now Z_1 is connected directly * to junction point B instead of A. Such a "transfer of the terminal" will always take place, for instance, when the two leads crossing each other contain only one element each—instead of the two assumed in figure 6.

However, the transfer method is not restricted to the presence of such crossing branches. Quite generally if (figure 9 a) A and B represent two junction points of the network, and Z_t an element fed from point A, we can join A to B

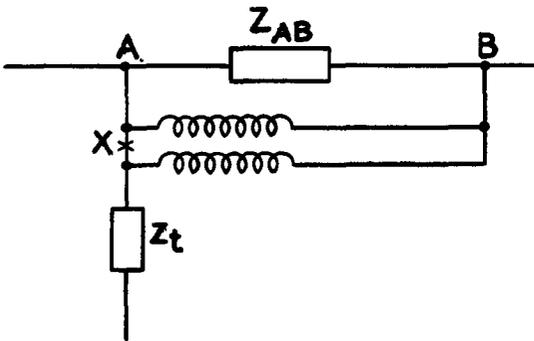


Figure 9 a.

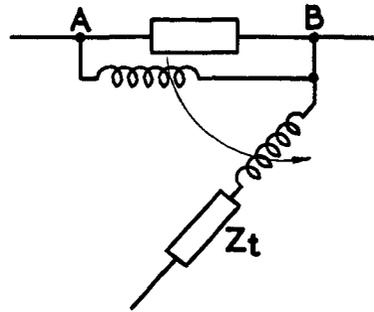


Figure 9 b.

by the primary and secondary of an ideal transformer, as shown. We are then allowed to break the connection between primary and secondary at X—for on account of the transformer action no difference of potential can arise between the separated leads and, therefore, no change in the current through Z_t can take place. This current is, however, now supplied from B instead of from A; in addition, there flows in the primary of the transformer a current equal to that taken from B, in the direction from A to B. Hence, as far as the rest of the network is concerned, the current distribution remains unchanged. For the sake of clarity the change in the configuration of the network brought about by this step is illustrated again in figure 9 b.

* It is true that in the diagram shown, the transformer coil is interposed between the terminal of B and A; however, the diagram could obviously have been drawn with the order of these two elements in branch 1 inverted.

(f) *The general method of separation*

This method, which can be described as a generalization of the dual to Julia's method, is illustrated in figure 10. Instead of eliminating a single branch from the original network we eliminate at one time several branches which together form one or more subsidiary networks.

Suppose we select in the original network a closed mesh containing the junction points (in this order) $MNPQ \dots$ and we join M to N , by the primary of an ideal two-winding transformer, similarly N to P , and so on. We are then allowed to join consecutively all the secondaries of these transformers so as to form a closed mesh. The mesh of the secondaries forms a kind of "parallel" mesh to the mesh formed by the primaries. Let us label the junction points of this second mesh $M'N'P'Q' \dots$. It is obvious that the potential difference between any one pair of these latter points is a replica of the potential difference of the corresponding unprimed points. For this reason any branch extending

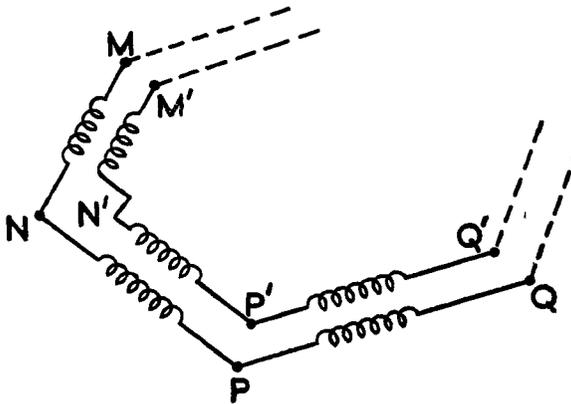


Figure 10.

between any pair of the unprimed junction points can be transferred to the corresponding primed points on the parallel mesh without altering the conditions of its working. The current drawn by such a branch still leaves the original network at the old junction points as currents which feed the primaries of the transformers concerned. Hence we have here a very simple means by which we can split the original network into two subsidiary networks which are only coupled by transformer action.

It is easily seen that the transformers need not actually form a closed mesh: one of the sides of the mesh may be left out. Furthermore, the mesh selected may—as was the case before (figure 5 a)—contain branches of infinite impedance.

This concludes the description of the various methods of network transformation which, it is felt, covers all needs arising in practice. No attempt has been made to cover the subject exhaustively.

Application

The possibility of constructing without restriction a network that is physically dual to a given network is in itself only of academic interest. It may, however,

become of immediate practical interest in connection with the increasing use of electro-mechanical analogies. As shown elsewhere (e.g. Bloch, 1945), we can study the action of an electro-mechanical system if we connect the electrical part of this system to a network which is the electrical analogue of the mechanical part of this system; this reduces the analysis to that of a single electrical network. However, this possibility exists only when we use the correct type of analogy: the direct one for the electrostatic type of coupling element and the inverse one for the electromagnetic type of coupling element. For the reasons already mentioned in the introduction to this paper, this method would break down—but for the methods here described—if we had to use the direct analogy in a case where the mechanical layout, and therefore the inverse analogue, were non-planar.

The methods of this paper enable us

(a) To draw a diagram of the direct analogue, and hence of the combined network, which we can use as a basis for further calculation or consideration. In this case it is obvious that we need not worry about the performance of those ideal transformers; they are ideal by definition.

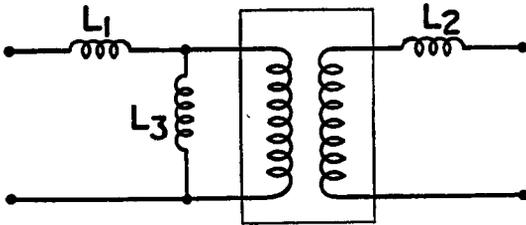


Figure 11 a.

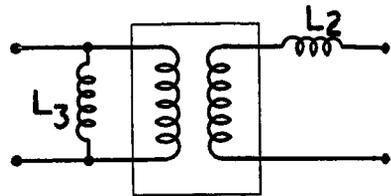


Figure 11 b.

(b) To set up an actual circuit, say, on a network analyser which will enable us to study directly by measurement the performance of the electrical circuit or of the underlying mechanical system. In this case the ideal transformers have to be replaced by real ones. As real transformers do not behave exactly like ideal ones, errors may arise from this, and it is here that the possibility of choosing from a large number of different methods becomes of value; it may be possible then to select a method in which the imperfection of the real transformers are innocuous.

Figure 11 (a) shows a well-known equivalent circuit for a real transformer, i.e. a theoretical conception which explains the behaviour of a real transformer as arising from the combination of an ideal transformer with certain inductances: leakage inductances L_1 and L_2 , and shunt inductances L_3 . It is perhaps worth while to point out that the distribution of the leakage inductance on the primary and secondary side of the transformer is to some extent arbitrary, and depends on the assumption made as to the magnitude of the ratio of the ideal transformer. By a suitable assumption it is possible to have all the leakage inductance concentrated on the primary or on the secondary side. We arrive thus at an equivalent circuit such as shown in figure 11 b.

The error caused by the leakage inductance is of course small in all those

cases where the impedance of this additional inductance is small compared with all the other impedances in series with it. Similarly the error caused by the presence of the shunt inductance will be small when its admittance is small compared with the other admittances across which it is connected. In certain cases it will be possible to reduce these errors actually to zero—at least as far as the rest of the network is concerned. Suppose the transformer coil is connected in series with another inductance larger than the leakage inductance. Obviously we need then only reduce this other inductance by the amount of the leakage inductance in order to arrive at a combination which to the rest of the network

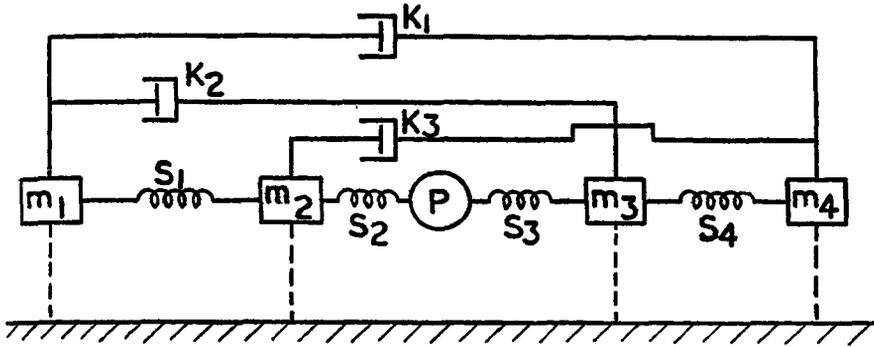


Figure 12.

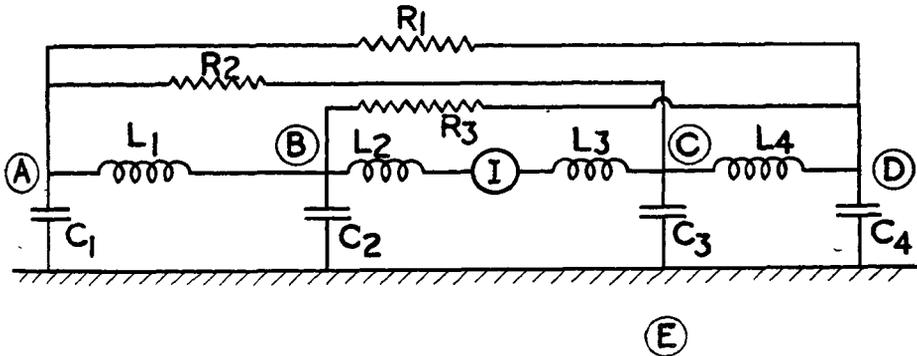


Figure 13.

looks exactly like the ideal combination. Similarly we can deal with the case of a shunt inductance in parallel to another inductance. The equivalent circuit of figure 11 *b*, in which all the leakage inductance is concentrated on one side, is specially useful for dealing with such cases. The combinations just referred to are wanted in the physical dual circuit; this means that when planning the original equivalent planar network we have to look ahead for their dual counterparts. Hence, for instance, when we wish to eliminate entirely the error caused by the leakage inductance we have to look for an equivalent network in which one transformer coil is placed in parallel to a capacity of sufficient magnitude.

Similarly, we have to place one transformer coil in series with a capacitive impedance of sufficient magnitude when we finally wish to eliminate the error caused by the shunt inductance.

An application of the aforesaid principles and of the principles of this paper in general is illustrated in figures 12 to 14. Figure 12 depicts a mechanical system consisting of a prime mover and of four masses which are inter-linked by an arrangement of springs and viscous couplings. The mode of presentation follows the convention which facilitates the drawing of the circuit of the electrical

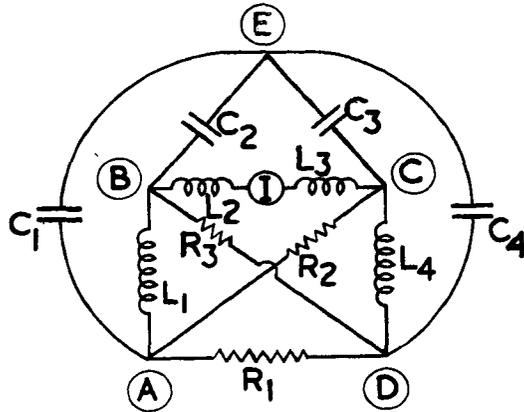


Figure 14.

inverse analogue. This circuit is shown in figure 13, and redrawn in a somewhat more familiar form in figure 14. From the latter figure it can be seen immediately that, for instance, a transformer could be placed in parallel to C_2 , and thereby the terminal of C_1 transferred from E to B. Other "parallel" solutions are also obvious. A number of "series" solutions has already been given in figures 8a) to 8c); some of these (e.g., 8b or 8c) enable us to place one transformer coil in series with a capacity—as stipulated above for the compensation of shunt inductance in the dual network

REFERENCES

- BARNES and GARDENER, 1942. *Transients in Linear Systems* (John Wiley), p. 49.
 BLOCH, A., 1945. *J. Inst. Electr. Engrs*, **92**, Part 1, 157.
 CAUER, W., 1934. *Z. angewandte Math. Mech.*, **14**, 349.
 JULIA, R., 1939. *Bull. Soc. Franç. Electr.*, **9**, 28.
 RUSSELL, A., 1904. *Alternating Currents*, chapter XVII (Cambridge University Press).
 SCHUR, F., 1897. *Math. Annal.*, **48**, 142.
 SIRE DE VILAR, H., 1901. *L'Eclairage Electrique*, **27**, 252.
 WHITNEY, HASSLER, 1931. *Proc. Nat. Acad. Sci.*, **17**, 125.